REMARKS ON THE TOPOLOGY OF THE FANO SURFACE

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Abstract. We analize the semistable degeneration of the smooth Fano surface F when the cubic threefold becomes the Segre primal. This gives an explicit topological decomposition for F. The components are surfaces with boundary, they come from two of the simplest line arrangements in the projective plane. The combinatorics of their intersections is described by a Kneser graph. The decomposition is used to decide that the Fano surface is not an an Eilenberg Mac-Lane $K(\pi, 1)$ space, this was the question that prompted us to look into the matter.

1. Introduction

Consider a cubic threefold $X \subset \mathbb{P}^4$, the Fano variety $F(X)$ is the locus inside the Grassmannian $G(1,4)$ made of the lines on X. If X is smooth then $F(X)$ is a non singular surface whose Hodge structure is classically well understood [2]. We provide an explicit topological decomposition for $F(X)$ which we use to show that the surface is not an an Eilenberg Mac-Lane $K(\pi, 1)$ space, in fact $\pi_2(F)$ is not of torsion. This gives a negative answer to a question asked by P.Pirola as an afterthought to [7]. Our result should be compared with Roulleau's ones, he proved in [15] that there are ball quotients sitting as Zariski open subsets in the Fano surfaces of the Fermat cubics. . The decomposition that we use is obtained by means of a semistable degeneration of the Fano surface, one which is associated with the degeneration of the cubic threefold when it moves in a one-parameter family and specializes to the Segre primal. This threefold has ten nodes, the largest number of isolated singularities that a cubic threefold can have. The degeneration for the Fano surface is also maximal, the singular fibre splits in 21 components, all rational surfaces. The nature of the components and the combinatorics of their intersections is dictated by the geometry of $\overline{M}_{0,6}$ (the moduli spaces of stable 6-pointed rational curves), this is so because $\overline{M}_{0,6}$ is a natural desingularization of the Segre cubic. The limiting Mixed Hodge structure on H^1 of a nonsingular fiber F_t is totally generated, i.e. it is of Hodge-Tate type. The Clemens-Schmid exact sequence can then be used to prove a certain injectivity statement. This is the main tool in our proof that $\pi_2(F)$ is not of torsion, given in the last section. Although the statement is purely topological it is apparent that our argument depends crucially on fundamental theorems of Hodge theory, cf. [10].

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2. A topological decomposition of the Fano surface

2.1. **The degeneration method.**

. The Fano surface F of a smooth cubic 3-folds is non singular and then any two of them are homeomorphic. The methods of [2] and [3], see also [14], work quite effectively for a pencil of cubic 3-folds with central fibre the Segre primal, the outcome is an explicit topological decomposition for F .

2.1.1. *The general setting of the degeneration method for surfaces.*

. Consider $q : \mathcal{Y} \to \Delta$, a proper flat holomorphic map from a smooth, analytic space to the unit disk. We say that this is a degeneration of surfaces if $Y_t := q^{-1}(t)$ is a complete variety, of dimension 2 and moreover Y_t is non singular for $t \neq 0$. The degeneration is stable if the central fibre Y_0 is a reduced divisor with global normal crossing singularities. A stable degeneration contains rich information relating the singular and the smooth fibre, our immediate aim is to recall the surgery procedure. This is a gluing process which builds topologically the smooth fibre by means of the decomposition in irreducible components of the singular fibre.

. Let $\mathcal{D} \subset Y_0$ be the singular divisor on the central fibre, it is the union of the double curves, where two components meet, and let $\mathcal{T} \subset \mathcal{D}$ be the set of triple points, the locus of intersection of three components. The surgery process depends on the existence of Clemens' collapsing function $c : \mathcal{Y} \to Y_0$, which maps the total space onto the central fibre. The restriction $c_t : Y_t \to Y_0$ has the properties

(1) $c^{-1}(y) = \mathit{point}$, if $y \in Y_0 \setminus \mathcal{D}$
(2) $c^{-1}(y) = S^1$, if $y \in \mathcal{D} \setminus \mathcal{T}$
(3) $c^{-1}(y) = S^1 \times S^1$, if $y \in \mathcal{T}$.

The gluing takes place along the boundary manifolds.

2.1.2. *The open complement and the boundary manifold.*

. Let S be a non singular surface and $C := \cup C_j \subset S$ be a connected normal crossing divisor with non singular components. In our situation the components of C are going to be lines and they intersect is at most one point. Consider the complement

$$S^0(C) := S \setminus C \,,$$

take an open regular normal neighbourhood N^0 of C and look at the boundary

$$M(C, S) := \partial N^0 \,.$$

This M is our boundary manifold, of real dimension 3. Since M is the boundary also of

$$\bar{S}(C) := S \setminus N^0 \,, \tag{2.1}$$

which is homotopically equivalent to $S^0(C)$, then Lefschetz duality and excision yield:

$$H_1(S^0(C)) \simeq H_1(S \setminus N^0) \simeq H^3(S \setminus N^0, M) \simeq H^3(S, C) \,. \tag{2.2}$$

The fibre of the projection $g : M(C, S) \to C$ is S^1, for a smooth point of C, while it is homeomorphic to $S^1 \times S^1$ over a node, cf. [14]. More precisely $M(C, S)$ is homotopic to $(C \setminus T) \times S^1$ here T is the non empty set of double points of C. The torus comes from the restriction of the S^1 fibration to the boundary of a disc around a point in T. In [5] or [13] one finds detailed descriptions of the properties of boundary manifold for lines arrangements. The surfaces that appear in our decomposition of F come from the simplest instances of arrangements, they are the complement B^0 of four general lines in $\mathbb{P}^2$ and the complement D^0 of the ten -1 lines in the del Pezzo surface of degree 5. One has

$$\pi_1(B^0) = H_1(B^0) = \mathbb{Z}^3 \;\text{ and }\; H_1(D^0) = \mathbb{Z}^5 \,. \tag{2.3}$$

2.1.3. *The topological decomposition.*
. The central fibre of our degeneration $\mathcal{Y}$ is a normal crossing divisor, $Y_0 = \cup Y_n$. On a component the double curve $D_n := \mathcal{D} \cap Y_n$ is singular precisely along the set of triple points $T_n := \mathcal{T} \cap V_n$. Consider now the manifolds $\bar{Y}_n(D_n)$ and their boundaries $M(D_n, Y_n)$. One has:

Theorem 2.1. [Clemens] *Y_t is homeomorphic to the topological space built by gluing the disjoint union of the manifolds $\bar{Y}_n(D_n)$ along their boundaries so that there is identification of fibres over the same point in $\mathcal{D}$ for the different projections $M(D_n, Y_n) \to D_n$.*

There are now two compatible Mayer-Vietoris spectral sequences on the stage, one abutting to $H^m(Y_0)$ and the second to $H^m(Y_t)$, see (2.5) in [14]; the contravariant morphism

$$c^* : H^m(Y_0) \to H^m(Y_t) \tag{2.4}$$

is induced by the natural map among them. In particular using (2.2) we note that for a component S of Y_0 as above, the map $H_1(S^0(D_S)) \to H_1(Y_t)$ can be factorized as

$$H_1(S^0(D_S)) \simeq H^3(S, D_S) \to H^3(Y_0) \to H^3(Y_t) \simeq H_1(Y_t) \tag{2.5}$$

In our special situation, the case of the degeneration of the Fano surface that we consider next, we shall check that both arrows $H^3(S, D_S) \to H^3(Y_0)$ and $H^3(Y_0) \to H^3(Y_t)$ are injective.

2.2. **The Segre primal and its Fano surface.**

2.2.1. *The Segre primal.*
. Consider the isomorphism of $\mathbb{P}^4$ with the diagonal hyperplane in $\mathbb{P}^5$ of equation $\sum x_i = 0$. The Segre primal $\mathcal{S}$ is the hypersurface in $\mathbb{P}^4$ given by the intersection of the hyperplane with the Fermat cubic fourfold. On $\mathcal{S}$ there 10 double points, and therefore one has also 45 distinguished lines, each one spanned by a couple of double points. It turns out that there are 15 planes on $\mathcal{S}$, any one of them passes through four of the nodes and contains six distinguished lines [16]. The geometry of $\mathcal{S}$ has been investigated recently in [11], there a proof of the non rationality of the generic smooth cubic threefold is given by making use of Alexeev's theory [1]. The proof depends on certain intriguing connections which relate the combinatorics of the configurations above with the classification of matroids.

2.2.2. *The Fano surface $F(\mathcal{S})$.*
. Our interest in the Segre primal is due to the structure of $F(\mathcal{S})$, the Fano surface of the lines which lie in $\mathcal{S}$. It turns out that $F(\mathcal{S})$ is made of 21 components. The description of what the components are, of the combinatorics of their intersection, and of the action of S_6 on $F(\mathcal{S})$ is easier to understand and simpler to state by making use of the property that $\mathcal{S}$ can be described as the image of the universal stable line with 5 marked points on it, which image is obtained by contracting the tails of the reducible curves.

2.2.3. *Reminder on moduli spaces of stable n-pointed rational curves.*
. A stable n-pointed rational curve is a connected curve C whose irreducible components are projective lines, with the property that the singularities are ordinary nodes and with the choice of n distinct marks which are smooth points of C, such that every irreducible component has at least three special points. Here special point means a mark or a node. The moduli space for stable n-pointed rational curves $\overline{M}_{0,n}$ as a set is the family of isomorphism classes of such curves. Mumford and Knudsen proved that it has the structure of a smooth irreducible projective variety of dimension $n-3$. Inside $\overline{M}_{0,n}$ we find as an open subset the moduli space of smooth stable irreducible pointed curves $M_{0,n}$. There is a stratification of $\overline{M}_{0,n} \setminus M_{0,n}$ by topological type of the dual graph of the curve: a *codimension 1-stratum* is an irreducible component of the locus

of points of $\overline{M}_{0,n}$ having at least one node. The generic element of a codimension 1-stratum has two irreducible components, with some $J \subseteq \{1, \dots, n\}$, $2 \le |J| \le n-2$, giving the marked points on one component, and J^c giving the marked points on the other. The resulting divisor is called a boundary divisor, and it is denoted by Δ_J. We write

$$\Delta := \partial \overline{M}_{0,5} = \cup \Delta_{i,j} \ , \ \text{so it is} \quad M_{0,5} = \overline{M}_{0,5} \setminus \Delta \, . \tag{2.6}$$

2.2.4. $\pi_1(M_{0,5})$.

. A presentation for $\pi_1(M_{0,5})$ was already known to Picard, cf. [19]. Let here x_1, x_2, x_3 be homogeneous coordinates for $\mathbb{P}^2$, set $x_4 = 0$, and write $S(ij) := \{x_i = x_j\} \subset \mathbb{P}^2$, $\{i,j\} \subset \{1..4\}$, $S(ijk) := \{x_i = x_j = x_k\} \subset \mathbb{P}^2$, $\{i,j,k\} \subset \{1..4\}$. The blow-up of $\mathbb{P}^2$ at the four points $S(ijk)$ gives a del Pezzo surface D of degree five. We have ten (-1) lines on D, which are $S(ij)^b$, the proper transform of $S(ij)$, and $S(ijk)^b$, the exceptional line over $S(ijk)$. It is well known that $D \simeq \overline{M}_{0,5}$, the identification is seen by realizing that D is the universal line over $\overline{M}_{0,4}$, the fibres being the transforms of the conics through the four base points $S(ijk)$. It is

$$S^b(ij) = \Delta_{k,l} \ , \quad \text{and} \quad S^b(ijk) = \Delta_{l,5} \quad \text{where } \{i,j,k,l\} = \{1..4\} \, .$$

The fundamental group $\pi_1(M_{0,5})$ is generated by ten loops, each one a normal loop around one of the ten -1 lines. Six of the loops can be written as $\sigma_{i,j}$ to indicate that they are the normal loops around $S^b(ij)$. The loop around the exceptional line $S^b(ijk)$ is then homotopic to the product $\sigma_{i,j}\sigma_{i,k}\sigma_{j,k}$. The commuting relations written in [19] say that a loop around an exceptional line commutes with the loops around the other lines which meet it. There is the further relation

$$\sigma_{12}\sigma_{13}\sigma_{23}\sigma_{14}\sigma_{24}\sigma_{34} = 1 \, . \tag{2.7}$$

We have therefore:

$$H_1(M_{0,5}, \mathbb{Z}) \simeq \mathbb{Z}^{\oplus 5}. \tag{2.8}$$

Remark 2.2. It is known that $M_{0,5}$ is a $K(\pi,1)$ space, as it can be seen using the long exact sequence of homotopy for the fibration $M_{0,5} \to M_{0,4}$.

2.2.5. *Irreducibility of the representation of* S_5 *on* $H^1(M_{0,5}, \mathbb{Q})$.

. The symmetric group S_5 acts on $\overline{M}_{0,5}$ and in particular it acts on the ten dimensional vector space B with basis $\Delta_{i,j}$. Let V be the standard 4 dimensional representation of S_5 sitting inside the tautological permutation representation T . Now $Sym^2 T \ \simeq T \oplus Sym^2 V$, where the copy of T comes from the squares. We have then $B \simeq Sym^2 V$, so by (3.2) in [9] it is

$$B = U \oplus V \oplus W \, , \tag{2.9}$$

here U is the trivial representation and W is irreducible of dimension 5.

We use this decomposition to determine the action of S_5 on $H^1(M_{0,5})$ and $H^2(\overline{M}_{0,5})$. The cohomology sequence with compact support of the couple $(\overline{M}_{0,5}, \Delta)$ is :

$$\cdots \to H^2(\overline{M}_{0,5}) \xrightarrow{j} H^2(\Delta) \to H^3_c(M_{0,5}) \to H^3(\overline{M}_{0,5}) \to \dots \ . \tag{2.10}$$

and dually , cf. [8], one has the sequence

$$\dots \leftarrow H^2(\overline{M}_{0,5}) \leftarrow H^2_\Delta(\overline{M}_{0,5}) \leftarrow H^1(M_{0,5}) \leftarrow \dots \ . \tag{2.11}$$

Our surface $\overline{M}_{0,5}$ is rational and the divisors $\Delta_{i,j}$ generate H^2 , then the last sequence yields:

$$0 \leftarrow H^2(\overline{M}_{0,5}) \leftarrow B \leftarrow H^1(M_{0,5}) \leftarrow 0 \ . \tag{2.12}$$

Both spaces $H^2(\overline{M}_{0,5})$ and $H^1(M_{0,5})$ are of dimension 5, but $H^2(\overline{M}_{0,5})$ contains an invariant one dimensional subspace, the one which is generated by the class of the canonical divisor. This proves

Proposition 2.3. *$H^1(M_{0,5}, \mathbb{Q}))$ is an irreducible representation of S_5 .*

2.2.6. *A birational map from $\overline{M}_{0,6}$ to the Segre primal $\mathcal{S}$.*
. The moduli space of stable 6-pointed rational curves can be mapped inside $\mathbb{P}^4$ birationally (this is a special case of a Kapranov's theorem) and the image is isomorphic to $\mathcal{S}$. This map $\overline{M}_{0,6} \to \mathcal{S}$ contracts the 10 boundary divisors Δ_J , $|J| = 3$ to the 10 nodes on $\mathcal{S}$, the 15 boundary divisors Δ_J , $|J| = 2$ are mapped to the 15 planes on $\mathcal{S}$.

Recall next the description of $\overline{M}_{0,6}$ as the universal curve. This amounts exactly to choose a label i and then to take the forgetful map $\phi_i : \overline{M}_{0,6} \to \overline{M}_{0,5}$ given by dropping the i^{th} marking. The contraction $\overline{M}_{0,6} \to \mathcal{S}$ maps the fibres of the universal curve to lines on $\mathcal{S}$.

2.2.7. *The projective construction of $\mathcal{S}$.*
. It is classically known that $\mathcal{S}$ is the image of the rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^4$ given by the linear system of quadric surfaces passing through five points p_i in general position. This fact is the key to understand Kapranov's contruction. So one blows up first $\mathbb{P}^3$ at the five points and then blows down the proper transforms ℓ_{ij} of the ten lines which join p_i and p_j. We write $p_{i,j}$ for the node on $\mathcal{S}$ image of ℓ_{ij}. Ten of the planes in $\mathcal{S}$ are the proper transforms of the planes in $\mathbb{P}^3$ containing three base points. The remaining five are the images of the exceptional divisors in the blowup. Each of the planes contains four nodes. Beside the lines contained in the planes, $\mathcal{S}$ contains six two dimensional families of lines, five of them are the families of the proper transforms of the lines through the chosen points, and one parametrizes the images of the twisted cubics through all of them.

2.2.8. *The components of $F(\mathcal{S})$.*
. This explains why the Fano surface $F(\mathcal{S})$ has 21 components, 15 of them are planes and 6 of them are del Pezzo surfaces of degree 5. The 6 del Pezzo surfaces, $D(i) \subset F(\mathcal{S})$, are the result of a different choice for the label i of the forgetful map $\phi_i : \overline{M}_{0,6} \to \overline{M}_{0,5}$ considered as the universal curve fibration. The 15 planes $P(j,k) \subset F(\mathcal{S})$ can be understood as the family of lines sitting inside each one of the 15 planes on $\mathcal{S}$, which planes on $\mathcal{S}$ are the images of the boundary divisors $\Delta_{i,j}$ on $\overline{M}_{0,6}$. The outcome is

$$\mathcal{F}(S) = (\bigcup_{i=1..6} D(i)) \bigcup (\bigcup_{\{j,k\}\subset\{1..6\}} P(j,k)\ ) \quad .$$

On $\mathcal{S}$ there are 45 distinguished lines, spanned by couple of nodes of $\mathcal{S}$. We write them $L[(i,j),(k,l)]$, which we identify with points in the Grassmann variety, i.e.

$$L[(i,j),(k,l)] := P(i,j) \cap P(k,l) \in F(\mathcal{S}) \subset G(1,4)\,.$$

A further consequence of the description of $\mathcal{S}$ as the image of $\overline{M}_{0,6}$ yields

$$P(j,k) \cap P(l,m) = \emptyset \ \text{ if } \{j,k\} \cap \{l,m\} \neq \emptyset.$$

With our notation $D(m)$ is a copy of $\overline{M}_{0,5}$, we write $R(m,[i,j])$ to represent the boundary divisor $\Delta_{i,j}$ on this copy of $\overline{M}_{0,5}$. One has

$$R(m,[j,k]) = D(m) \cap P(j,k) \quad , \text{if } m \notin \{j,k\}\,.$$

On the other hand

$$D(j) \cap P(j,k) = \emptyset\,.$$

. On each curve $R(m,[j,k])$ there are three distinguished points ,

$$L[(j,k),(i,n)] \in R(m,[j,k])\,,\ \text{if}\ \{i,n\}\cap\{m,j,k\}=\emptyset\,.$$

The intersection of two del Pezzo components is the set

$$D(m)\cap D(n)=\{L[(i,j),(k,l)],L[(i,k),(j,l)],L[(i,l),(k,j)]\}_{\{m,n\}\cap\{i,j,k,l\}=\emptyset}$$

Finally

$$L[(i,j),(k,l)]=D(m)\cap D(n)\cap P(i,j)\cap P(k,l)\,.$$

This is exactly the reason why $F(\mathcal{S})$ is not a normal crossing surface, along these points there meet 4 surfaces, and moreover the surfaces split in two couples which intersect locally only at the point.

2.3. A semistable degeneration for the smooth Fano surface.

. The results of [11] are based on the study of the geometry of the degeneration of cubic threefolds to the singular Segre primal $\mathcal{S}$, we use the same specialization to understand the topology of the Fano surface. An affine pencil of cubic threefolds in $\mathbb{P}^4$ is a hypersurface $\mathcal{X}\subset\mathbb{P}^4\times\mathbb{A}^1$ determined by its equation $tE+G=0$, E and G are cubic polynomials. The corresponding family of Fano surfaces, $\mathcal{F}\subset G(1,4)\times\mathbb{A}^1$ has fibres F_t, the surface of lines on X_t. By taking G to be the equation of $\mathcal{S}$ (up to a change of coordinates one such is $G:=\sum x_k^3+\sum_{i\neq j}x_i^2x_j=0$) $\mathcal{F}$ is a family of surfaces with central fibre $F(\mathcal{S})$. This is not a semistable degeneration, because the central fibre has not normal crossing and also because $\mathcal{F}$ is singular. We are concerned with the singularities of $\mathcal{F}$ just locally, near $t=0$.

It is known that the Fano surface of a cubic $3-$fold can be singular only along the locus of lines which meet the singular points of the cubic $3-$fold. As a consequence of this (by taking further the polynomial E general enough, so that X_t does not pass through the nodes of $\mathcal{S}$) we see that our threefold $\mathcal{F}$ can be singular only on the central fibre, and more precisely just along those points which represent lines on $\mathcal{S}$ passing through the nodes. Using Maple I have performed a computation akin to some which can be found in [2]:

Proposition 2.4. *Near $t=0$ the pencil $\mathcal{F}$ is singular exactly at the distinguished points $L[(i,j),(k,l)]$ of the central fibre $F(\mathcal{S})$. All those points are ordinary quadratic singularities.*

Our next step is to desingularize $\mathcal{F}$. One way is to blow up all the nodes, thus replacing each one of them with a non singular quadric surface. We find more convenient to choose instead the local analytic process of replacing every node with a projective line, thus performing 45 small blow-ups over $\mathcal{F}$. As it is well known there are two possible choices for each line. We take the one which has the effect to blow up the two planes containing $L[(i,j),(k,l)]$ at that point, and on the contrary leaves unchanged the two del Pezzo surfaces through it. Our concern here is in the topology of the Fano surface, but later we need to appeal to the Clemens Schmid exact sequence, and thus we need to check that the process of blow up described takes place in the Kaehler category. This is in fact the case, because our process can be performed by blowing up in a sequence the smooth del Pezzo surfaces inside the larger smooth ambient space on which the threefold lies. Each blowing up leaves unchanged the base surface and also the threefold away from the double points on the surface, while at those double points the node is blown up to a projective line, which becomes in our case an exceptional line on the proper transform on the planes passing through it. The other del Pezzo surfaces passing through one of the nodes of the base surface become disjoint from it, meeting the exceptional line in a different point from the isomorphic transform of the chosen base surface.

Definition 2.5. $\widetilde{\mathcal{F}}\to\mathcal{F}$ is the 45-small blow-up map, with the said requirements .

Proposition 2.6. *Locally near 0 on $\mathbb{A}^1$, $\widetilde{\mathcal{F}} \to \mathbb{A}^1$ is a semistable degeneration of surfaces, namely $\widetilde{\mathcal{F}}$ is non singular and the central fibre is union of smooth surfaces which have normal crossings. Recall that this says:*

1. *two components are either disjoint, or intersect transversally along a smooth irreducible curve.*
2. *three components meet each other at most in one point, and in that case analytically like three planes do.*
3. *four components have an empty intersection.*

Proof. It is an easy consequence of the combinatorics of intersection for the components of $F(\mathcal{S})$. □

2.3.1. *Combinatorics of the central fibre on $\widetilde{\mathcal{F}}$.*
. The threefold $\widetilde{\mathcal{F}}$ is the result of blowing up $\widetilde{\mathcal{F}}$ in such a manner that every plane $P(i,j)$ has been blown up at the 6 points $L[(i,j),(k,l)]$ which are the intersection of the four lines $R(m,[i,j])$:

Definition 2.7.

1. $\widetilde{F_0}$ is the central fibre of $\widetilde{\mathcal{F}}$
2. $B(i,j) \to P(i,j)$ is the blow-up of $P(i,j)$ at the 6 points $L[(i,j),(k,l)]$,
3. $E[(i,j),(k,l)] = \emptyset, \quad \text{if } \{i,j\} \cap \{k,l\} \neq \emptyset$
4. $E[(i,j),(k,l)]$ is the exceptional line over $L[(i,j),(k,l)]$.
5. $D(m)$ is the component on $\widetilde{F_0}$ which maps isomorphically to the surface by the same name on $F(\mathcal{S})$.
6. $R(m,[m,k]) = \emptyset$.
7. $R(m,[j,k])$ is the curve on $\widetilde{F_0}$ which maps isomorphically to the curve by the same name on $F(\mathcal{S})$.
8. $N(m,[(m,j),(k,l)]) := \emptyset$
9. $N(m,[(i,j),(k,l)]) := E[(i,j),(k,l)] \cap D(m) \subset \widetilde{F_0}$

With these conventions we can describe easily the configuration of the central fibre:

Proposition 2.8. *The central fibre has normal crossing components:*

$$\widetilde{F_0} = \Big(\bigcup_{i=1..6} D(i)\Big) \bigcup \Big(\bigcup_{\{j,k\}\subset\{1..6\}} B(j,k)\Big)$$

their intersections are:

- $D(m) \cap D(n) = \emptyset$.
- $B(m,n) \cap B(k,l) = E[(m,n),(k,l)]$.
- $D(m) \cap B(j,k) = R(m,[j,k])$.
- $D(m) \cap B(k,l) \cap B(i,j) = N(m,[(i,j),(k,l)])$.

2.3.2. *Kneser graphs and the dual complex of $\tilde{F}_0$.*
. The intersection pattern of the components of $\tilde{F}_0$ is codified by its dual simplicial complex $\Lambda(\tilde{F}_0)$, its vertices correspond to the components, two vertices are joined iff the components meet, and a $2-$simplex is formed out of 3 vertices iff the relative components intersect in a triple point.

. By definition the Kneser graph $KG(m,n)$, $m \geq 2n$, has vertex set the collection of all n-subsets of $[m]$, and two vertices are adjacent if and only if they are disjoint as n-subsets. The incidence graph of the lines on the Del Pezzo surface is $KG(5,2)$, otherwise known as the Petersen graph. Similarly the combination pattern of the surfaces $B(m,n)$ determines a graph Γ with vertices $v_{(m,n)}$ and then $\Gamma = KG(6,2)$.

. The dual simplicial complex $\Lambda(\tilde{F_0})$ has its $1-$skeleton isomorphic with $KG(7,2)$. The 6 vertices of type $[i,7]$ in $KG(7,2)$ correspond to the components $D(i)$, and the other vertices correspond to the components $B(m,n)$. The $2-$faces to be added in order to pass from $KG(7,2)$ to Λ are the triangles with a vertex of type $[i,7]$. A Gap computation, which was kindly shown to me by Prof. Soicher [17], says

Proposition 2.9. $\pi_1(\Lambda(\tilde{F_0})) \simeq \mathbb{Z}^{\oplus 5}$ *and therefore* $h_1(\Lambda(\tilde{F_0})) = 5$, $h_2(\Lambda(\tilde{F_0})) = 10$.

3. Some facts on the cohomology of $\tilde{F_0}$ and of F .

3.1. **The cohomology of $\tilde{F_0}$.**

. We recall next some fundamental results, see [10].

3.1.1. *Deligne's MHS on a normal crossing variety.*

. Let X be a simple normal crossing variety, so $X = \cup_i X_i$ where the X_i intersect locally like a union of coordinate hyperplanes. For $I := \{i_0, \dots, i_p\}$, let $X_I := X_{i_0} \cap \dots \cap X_{i_p}$ and let $X^{[m]} = \sqcup_{\{|J|=m\}} X_J$ be the disjoint union of the iterated intersections of lenght m . The MHS on the cohomology of X is determined by means of the spectral sequence which abuts to it. One has a double complex $E_0^{p,q} = A^q(X^{[p]})$ with vertical arrow the usual de Rham operator $d_0 = d$, while the horizontal arrow $d_1 = \delta$, is induced by alternating restriction of forms, so taking $I := \{i_0, \dots, i_{p+1}\}$ this is

$$(\delta\alpha)(X_I) := \sum (-1)^j \alpha(X_{I \setminus \{i_j\}})|X_I \ .$$

The first term of the spectral sequence is $E_1^{p,q} = H^q(X^{[p]})$ and then $E_2^{p,q}$ is the middle cohomology of

$$H^q(X^{[p-1]}) \xrightarrow{d_1} H^q(X^{[p]}) \xrightarrow{d_1} H^q(X^{[p+1]}) \tag{3.1}$$

The spectral sequence degenerates at E_2. The weight filtration for $H^m(X)$ is $W_\ell := \bigoplus_{s \leq \ell} E_2^{m-s,s}$, and the Hodge filtration is the filtration induced by the usual Hodge structure on each factor in E_1.

3.1.2. *Clemens-Schmid.*

. The Clemens-Schmid exact sequence for a semistable degeneration $\mathcal{X} \supset X_0$ is the exact sequence of MHS's:

$$\cdots \to H_{2n+2-m}(X_0) \xrightarrow{j} H^m(X_0) \xrightarrow{c^*} H^m \xrightarrow{N} H^m \xrightarrow{k} H_{2n-m}(X_0) \to \cdots \tag{3.2}$$

here N is the nilpotent monodromy operator and c^* as in (2.4). We take it in the form

$$0 \xrightarrow{j} H^1(\tilde{F_0}) \xrightarrow{c^*} H^1 \xrightarrow{N} H^1 \xrightarrow{k} H_3(\tilde{F_0}) \xrightarrow{j} H^3(\tilde{F_0}) \xrightarrow{c^*} H^3 \xrightarrow{N} H^3 \xrightarrow{k} H_1(\tilde{F_0}) \to 0 \tag{3.3}$$

so that H^m is the rational cohomology of the non singular Fano surface, equipped with the asymptotic mixed Hodge structure. It is $h_3(F) = h_1(F) = 10$.

. We need to understand some maps with range $H_1(F)$, which by Poincare duality is just $H^3(F)$. Our first aim is to see that $H^3(\tilde{F}_0) \xrightarrow{c^*} H^3 = H_1$ is an injection with image an isotropic subspace. We use here the intersection form on $H_1(F) = H^3(F)$ which is induced by the isomorphism $H_1(F) \simeq H_3(X) \simeq H_1(J)$, X being the smooth cubic 3-fold and J its polarized intermediate Jacobian, see the introduction in [2].

3.2. $H^3(\tilde{F}_0)$.

. The irreducible components of $\tilde{F}_0$ have dimension 2, so in the spectral sequence it is $E_1^{p,q} = 0$ for $p \geq 3$ and moreover one has $E_1^{2,q} = 0$ when $q \geq 1$. In our setting the components are rational surfaces which gives $E_1^{0,3} = 0$ and then $H^3(\tilde{F}_0) = E_2^{1,2}$, i.e. the following is an exact sequence:

$$H^2(\tilde{F}^{[0]}) \xrightarrow{d_1^{0,2}} H^2(\tilde{F}^{[1]}) \to H^3(\tilde{F}_0) \to 0 \tag{3.4}$$

Consider a component S of $\tilde{F}^{[0]}$; in S there is the normal crossing divisor C which is the intersection of S with the remaining components. As S is a rational surface then the long exact sequence of relative cohomology for the couple (S, C) reads

$$\cdots \to H^2(S) \xrightarrow{i_C^*} H^2(C) \to H^3(S, C) \to 0 \ . \tag{3.5}$$

It is clear that i_C^* coincides, up to sign, with the restriction of $d_1^{0,2}$ to S.
. Recall now the notations from (2.2), then the factorization (2.5) reads

$$H_1(S \setminus C) = H^3(S, C) \to E_2^{1,2} = H^3(\tilde{F}_0) \to H^3(F) = H_1(F) \tag{3.6}$$

Proposition 3.1.

(1) $dim_{\mathbb{Q}} H^3(\tilde{F}_0, \mathbb{Q}) = 5$.
(2) $H^3(\tilde{F}_0) \xrightarrow{c^*} H^3 = H_1$ *is an inclusion.*
(3) $H^3(\tilde{F}_0)$ *is a totally isotropic subspace of* H_1 .
(4) $H^3(J)$ *is the dual space of* $\wedge^3 H_1(F)$. *Under their pairing the unprimitive cohomology* $\theta \cdot H^1(J)$ *vanishes on the subspace* $\wedge^3 H^3(\tilde{F}_0)$, *and therefore for any nontrivial element in* $\wedge^3 H^3(\tilde{F}_0)$ *there is primitive class inside* $H^3(J)$ *which does not vanish on said element.*

Proof. The dual graph of the singular fibre is $\Lambda(\tilde{F}_0)$, whose betti numbers we have found in 2.9. In our setting one has $W_0 H^1(\tilde{F}_0) = W_1 H^1(\tilde{F}_0) = H^1(\tilde{F}_0)$, and therefore $dim H^1(\tilde{F}_0) = h^1(\Lambda(\tilde{F}_0)) = 5$. Now Lemma (2.7.4) in [14] yields that it is also $dim E_2^{1,2} = 5$, which is the first item, because $H^3(\tilde{F}_0) = E_2^{1,2}$. Statement (2) descends easily from exactness of 3.3. Items (3) and (4) are proved by recalling the identifications $H^3(F) \simeq H^3(X) \simeq H_3(X)$. By exactness $H^3(\tilde{F}_0, \mathbb{Q}) = KerN$, which is the module of invariant cycles for $H_3(X)$ and it follows from (1) that it coincides then with the modulo of vanishing cycles, which space is totally isotropic for the intersection pairing, cf. (3.3) in [4]. The proof of item 4 is routine, because to give the polarization θ on J is equivalent to give to the intersection coupling Q on $H_1(J) = H_3(X)$ by setting $(\theta, \alpha \wedge \beta) = Q(\alpha, \beta)$. □

Proposition 3.2. *(a) For a component* S *of type* $B(i,j)$ *the arrow*

$$H_1(S \setminus C) = H^3(S, C) \to E_2^{1,2} = H^3(\tilde{F}_0)$$

is an inclusion. (b) When the component S *is a Del Pezzo surface the same map gives an isomorphism:*

$$H_1(M_{0,5}) \simeq H^3(\tilde{F}_0)$$

Proof. To prove this directly by means of the spectral sequence requires dealing with large matrices, although of a simple type, which seems to be a rather complex task. We choose instead to use Prop. 2.3. The kernel of $H_1(M_{0,5}) \to H^3(\tilde{F}_0)$ is invariant under the action of S_5, if this map is not injective then it must be the trivial morphism. In this case, again by symmetry, the arrows above are then trivial for all the components which are Del Pezzo surfaces. On the other hand, having fixed i and j, recall that then $S_4 \subset S_6$ acts on the surface $B(i,j)$. By the same kind of argument as the one given for Prop. 2.3 we see that the representation of S_4 on $H^3(B(i,j), \partial B(i,j))$ is the standard representation, and therefore if $H^3(B(i,j), \partial B(i,j)) \to H^3(\tilde{F}_0)$ is not an injective morphism then it vanishes identically. Again by symmetry, if this was the case then the maps $H^3(B(i,j), \partial B(i,j)) \to H^3(\tilde{F}_0)$ vanish for all surfaces $B(i,j)$. Now one can check that a Del Pezzo component and one component $B(i,j)$ which intersect contribute at least a common generator to $H^3(\tilde{F}_0)$, the class of their intersection. In this way, if the map to $H^3(\tilde{F}_0)$ was not injective for one of this surfaces we know that it vanishes identically, and then also the map for the other surface should vanish on the common generator, hence identically. Looking at the boundary morphism in (3.4) we see that it is then $H^3(\tilde{F}_0, \mathbb{Q}) = 0$, which is a contradiction. □

4. On the second homotopy group of the Fano surface

4.1. **Exact Sequences.**

. For an abelian variety $\pi_1(A) = H_1(A, \mathbb{Z})$ and $\pi_i(A) = 0, i \geq 2$, and therefore the Albanese map $F \to A$, which is for the Fano surface an embedding, yields the diagram:

$$
\begin{tikzcd}
\pi_3(F) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \pi_3(A,F) \arrow[r] \arrow[d, "h_3"] & \pi_2(F) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \pi_2(A,F) \arrow[r] \arrow[d, "h_2"] & \pi_1(F) \arrow[r] \arrow[d] & \pi_1(A) \arrow[d, "="] \\
H_3(F) \arrow[r] & H_3(A) \arrow[r] & H_3(A,F) \arrow[r] & H_2(F) \arrow[r] & H_2(A) \arrow[r] & H_2(A,F) \arrow[r] & H_1(F) \arrow[r, "="] & H_1(A)
\end{tikzcd} \tag{4.1}
$$

We have $H_1(F, \mathbb{Z}) = H_1(A, \mathbb{Z}) \simeq \mathbb{Z}^{10}$, [2] Corollary 9.5. The generalized Hurewicz theorem, cf. [18], gives the surjectivity of h_2, because $\pi_1(A, F)$ is trivial, and therefore h_2 is the quotient map

$$[\pi_1(F), \pi_1(F)] \to [\pi_1(F), \pi_1(F)]/[[\pi_1(F), \pi_1(F)], \pi_1(F)] \, .$$

The lower central series of a group G being defined inductively as: $\Gamma_1 G = G$ and $\Gamma_{k+1} G = [\Gamma_k G, G]$, we see that $H_2(A, F)$ is the second graded piece of the lower central series for $\pi_1(F)$. One has, see [6],

$$H_2(A, \mathbb{Z})/H_2(F, \mathbb{Z})) \simeq \mathbb{Z}/2\mathbb{Z} \, . \tag{4.2}$$

Remark 4.1. It follows from 4.2 that the further graded quotients of the lower central series for $\pi_1(F)$ are all of exponent 2. I do not know if they vanish or not, nor I do know whether $\pi_1(F)$ is residually nilpotent, namely if the intersection of its lower central series is trivial. The statement in my paper [6] to the effect that $\pi_1(F)$ is a *central* extension of $H_1(F)$ with kernel $\mathbb{Z}/2$ is without proof. The commutator subgroup is indeed generated by an element of order 2 as a normal subgroup, but the further claim there used, that this is a central element, is unsubstantiated. It seems likely that one could give a procedure for computing the fundamental group by means of a detailed analysis of the topological decomposition. In this case implementation would then yield a list with the initial terms of the lower central series.

4.2. $\pi_2(F)$ **is not trivial.**

. We see from diagram 4.1 that in order to prove that $\pi_2(F) = \pi_3(A, F)$ is not trivial it is enough to determine an element which can be computed not to be in the kernel of the Hurewicz map h_3. To this aim we use the inclusion $B^0 \subset F$ where B^0 is homotopically equivalent to the complement of four general lines in the projective plane, cf. 2.8 above .
. According to [12] up to homotopy B^0 is the union of the 3 copies of $C^* \times C^*$ which inside $C^* \times C^* \times C^*$ are given by the condition that one of the coordinates is 1 and therefore B^0 is equivalent to the 2-skeleton of the real 3-torus T^3. One has

$$\pi_1(B^0) = H_1(B^0) = \mathbb{Z}^3 , \qquad H_2(B^0) = \mathbb{Z}^3 , \qquad H_m(B^0) = 0, m \geq 3 .$$

Consider the universal covers $\tilde{B}^0$ of B^0 and $\widetilde{T^3}$ of T^3:

Proposition 4.2. [12]

$$\pi_2(B^0) = H_2(\tilde{B}^0) \textit{ is a free } \mathbb{Z}\mathbb{Z}^3 \textit{ module } ,$$

generated by the boundary of a cubical 3*-cell from the standard decomposition of* $\widetilde{T^3} = \mathbb{R}^3$.

Our result is:

Theorem 4.3. $\pi_2(F)$ *is not of torsion.*

Proof. We work up to homotopy and use the map $B^0 \to \mathbb{C}^{*\,3}$.
. The Albanese variety A of F is isomorphic to the intermediate jacobian J of the corresponding cubic 3-fold, hence $A = J$ is a principally polarized abelian variety, of dimension 5 . The inclusions $B^0 \hookrightarrow F \hookrightarrow J$ induce the maps

$$\mathbb{Z}^3 = \pi_1(B^0) = H_1(B^0, \mathbb{Z}) \to H_1(F, \mathbb{Z}) = H_1(J, \mathbb{Z}) = \pi_1(J) = \mathbb{Z}^{10},$$

and therefore one has a continuous map

$$q : \mathbb{C}^{*\,3} \to J$$

which, up to homotopy, is a map of couples $q : (\mathbb{C}^{*\,3}, B^0) \to (J, B^0)$. There is a commutative diagram:
(4.3)

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[rr] \arrow[dr] \arrow[dd] & & \pi_3(\mathbb{C}^{*3}, B^0) \arrow[rr] \arrow[dr, "h_3"] \arrow[dd] & & \pi_2(B^0) \arrow[rr] \arrow[dr] \arrow[dd] & & 0 \arrow[dr] \arrow[dd] & \\
0 = H_3(B^0) \arrow[rr] \arrow[dd] & & H_3(\mathbb{C}^{*3}) = \mathbb{Z} \arrow[rr] \arrow[dd] & & H_3(\mathbb{C}^{*3}, B^0) \arrow[rr] \arrow[dd] & & H_2(B^0) = \mathbb{Z}^3 \arrow[rr] \arrow[dd] & & H_2(\mathbb{C}^{*3}) = \mathbb{Z}^3 \arrow[dd] \\
\cdots \arrow[r] & 0 \arrow[rr] \arrow[dr] & & \pi_3(J, F) \arrow[rr] \arrow[dr, "h"] & & \pi_2(F) \arrow[rr] \arrow[dr] & & 0 \arrow[dr] & \\
H_3(F) \arrow[rr] & & H_3(J) \arrow[rr] & & H_3(J, F) \arrow[rr, "0"] & & H_2(F) \arrow[rr] & & H_2(J)
\end{tikzcd}$$

It is $H_3(\mathbb{C}^{*\,3}, \mathbb{Z}) \simeq H_3(\mathbb{C}^{*\,3}, B^0)$ because $H_2(B^0, \mathbb{Z}) \simeq H_2(\mathbb{C}^{*\,3}, \mathbb{Z})$. The generalized theorem of Hurewicz for the couple $(\mathbb{C}^{*\,3}, B^0)$ says that the map $h_3 : \pi_3(\mathbb{C}^{*\,3}, B^0) \to H_3(\mathbb{C}^{*\,3}, B^0)$ is surjective and then the generator ζ in $H_3(\mathbb{C}^{*\,3})$ can be used to detect the existence of a non torsion element $z \in \pi_2(B^0)$. We map z to $\pi_2(F)$ i.e. to $\pi_3(J, F)$ then to an element $\bar{z} \in H_3(J, F)$. By commutativity $\bar{z}$ is the image of $q_*(\zeta) \in H_3(J)$, and then $\bar{z} \neq 0$ if $q_*(\zeta)$ does not come from $H_3(F)$. The results given above in 3.1 and 3.2 imply the existence of a primitive element in $H^3(J)$ which acts non trivially on $q_*(\zeta)$. This is precisely what we need to know, because the kernel of the restriction map $H^3(J) \to H^3(F)$ is formed by the primitive classes, cf. (0.9) in [2]. It follows that $\pi_2(F)$ is not of torsion. $\square$

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